Graphs and Flows

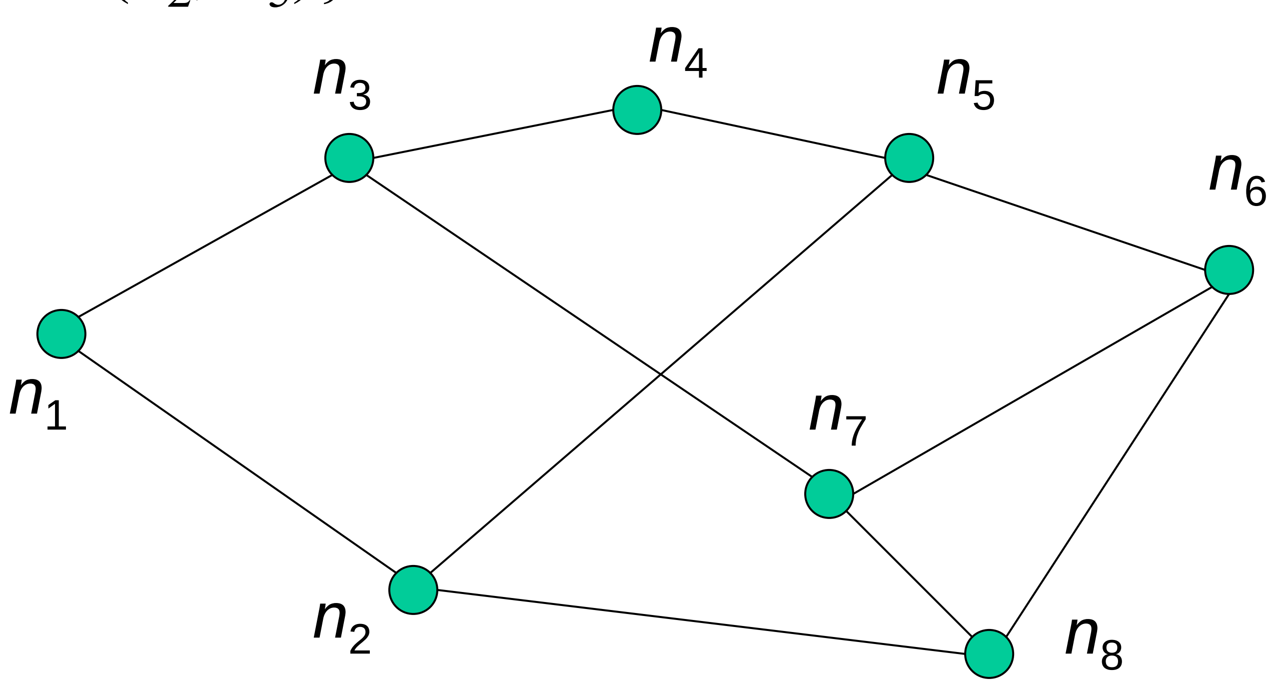
Paths and Cycles
Flow and Divergence
Path Flows and Conformal Decomposition

Reference

Dimitri Bertsekas, “Network Optimization” (http://www.athenasc.com/netbook.html)

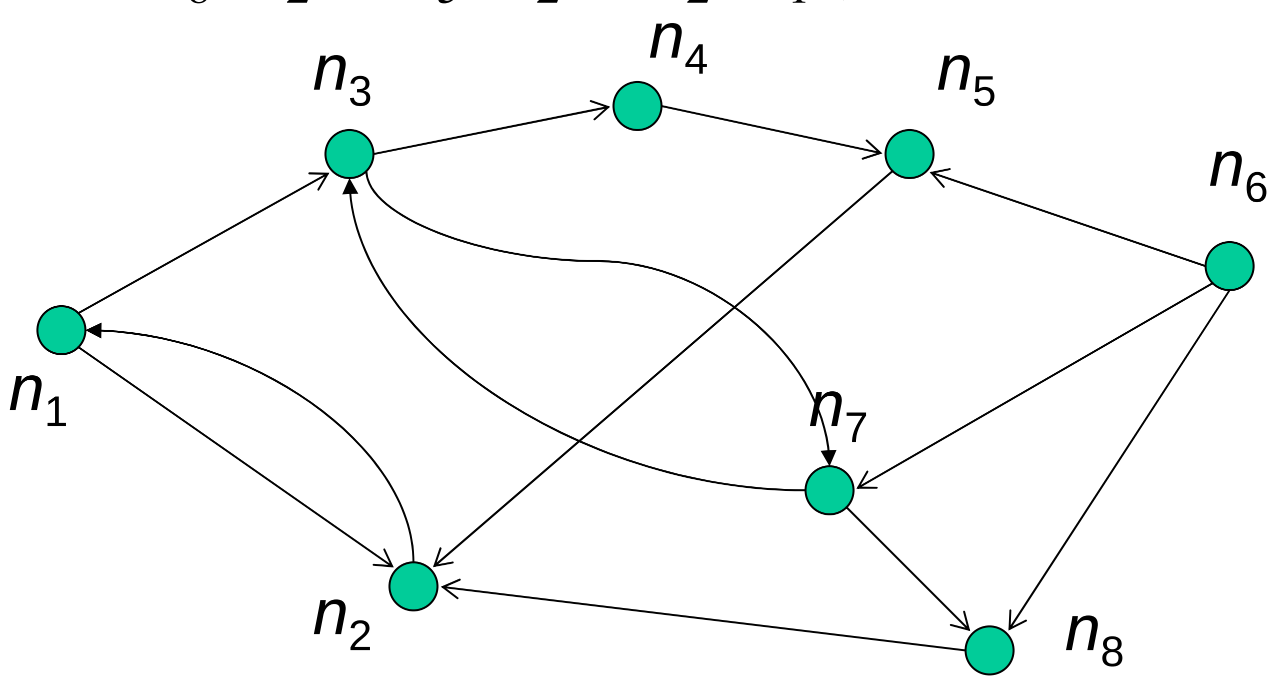
Graphs: Introduction

- Graph G=(N,A), N set of nodes, A set of pairs of distinct nodes from N called arcs.
- N={n_1, n_2,..., n_8}
- A={(n_1, n_2), (n_1, n_3), (n_3, n_4), (n_4, n_5), (n_5, n_6), (n_6, n_8), (n_6, n_7), (n_7, n_8), (n_3, n_7) , (n_2, n_8) , (n_2, n_5)}



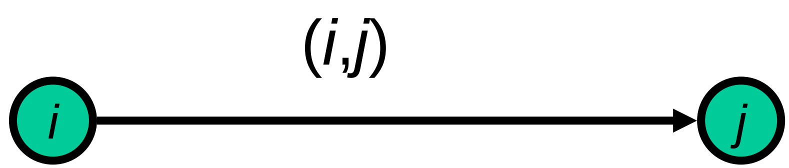
Directed Graph

- Graph G=(N,A), N set of nodes, A set of pairs of distinct nodes from N called arcs.
- $N=\{n_1, n_2, \ldots, n_8\}$
- A=$\{(n_1, n_2), (n_1, n_3), (n_3, n_4), (n_4, n_5), (n_6, n_5),$
  $(n_6, n_8), (n_6, n_7), (n_7, n_8), (n_3, n_7), (n_7, n_3),$
  $(n_8, n_2), (n_5, n_2), (n_2, n_1)\}$



Directed graphs: A terminology note

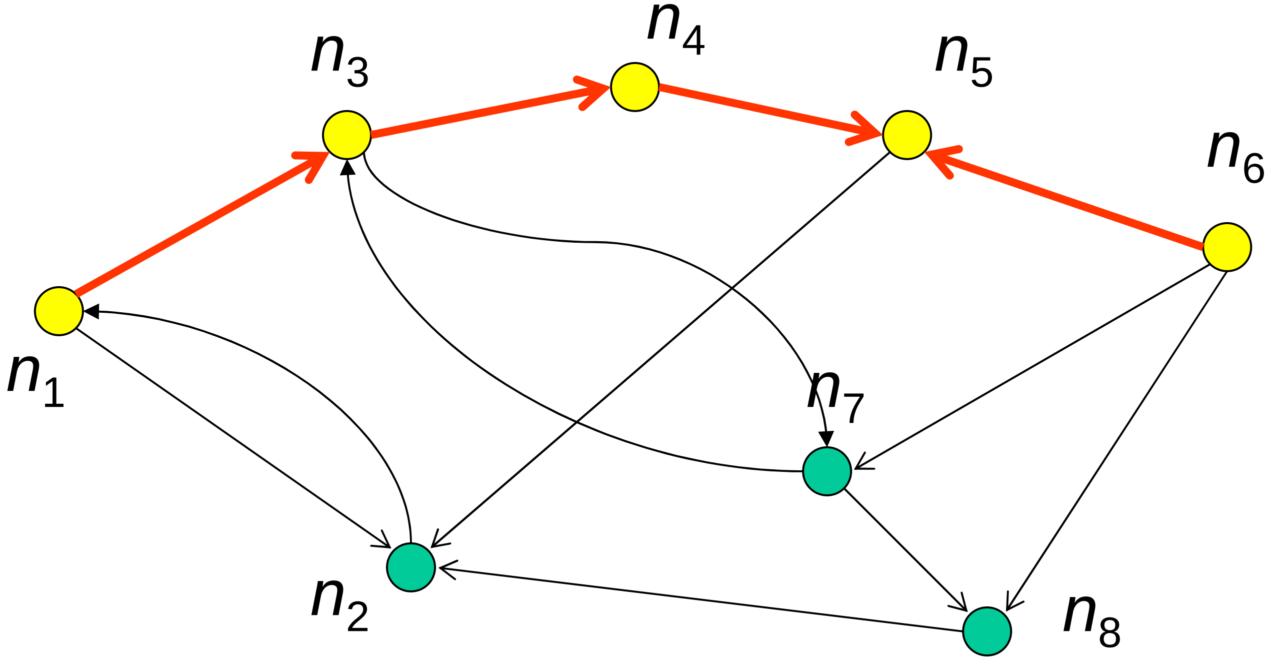
- Arc (i,j) “outgoing from node i”, “incoming to node j”; (i,j) “incident to i and j”, i the “start node”, j the “end node”.



- Directed graphs to be used throughout this chapter.

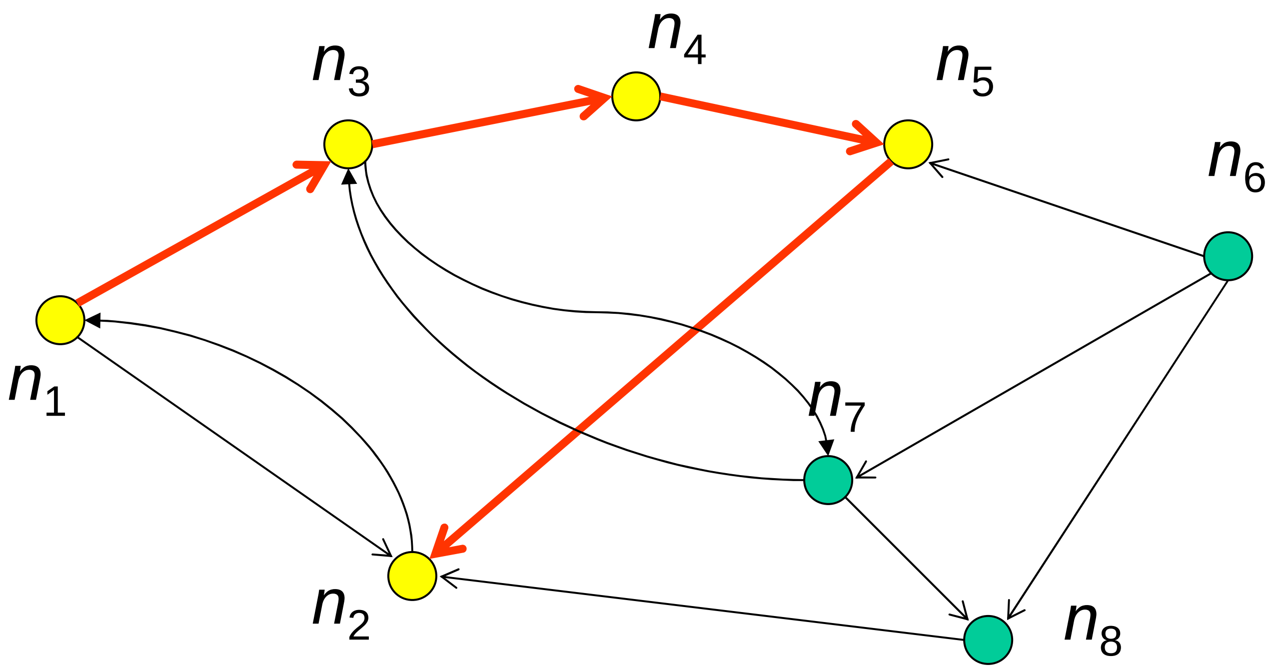
Paths

- *Path*: A sequence of nodes and a corresponding sequence of arcs, forward or backward.
- n_1 "start node", n_6 "end node".
- In path P, P^+ is the set of forward arcs, P^- is the set of backward arcs.



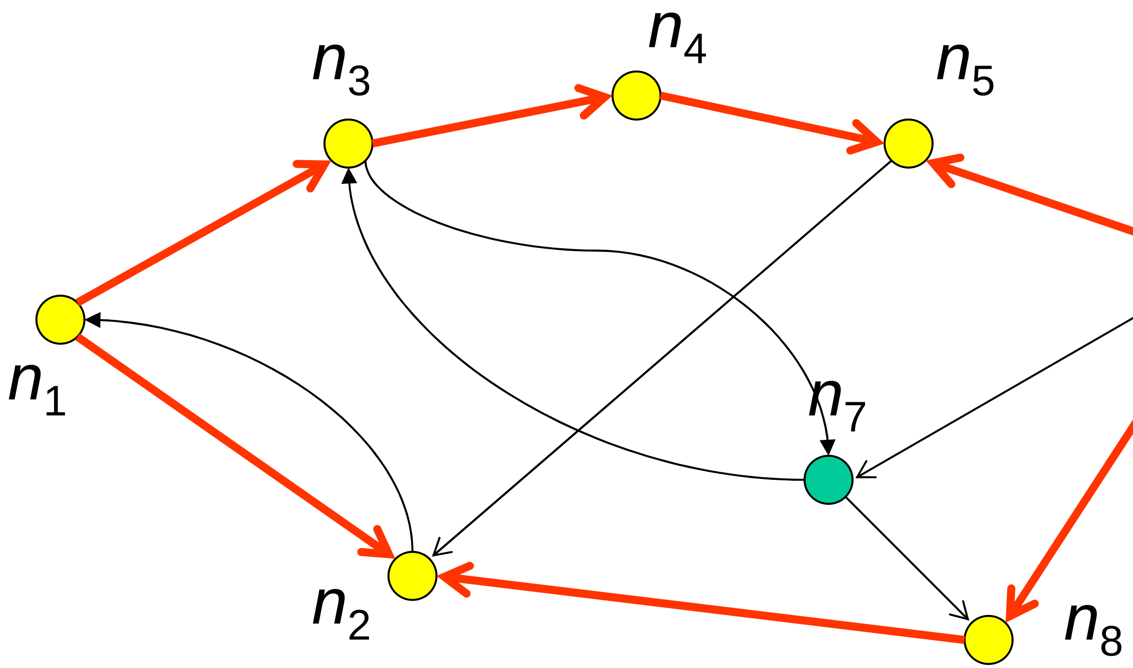
Paths

- *Forward (backward) path*: All arcs are forward (backward).
- *Simple path*: It contains no repeated arcs and no repeated nodes (except the start and the end node could be the same, then it is a *simple cycle*).



Paths

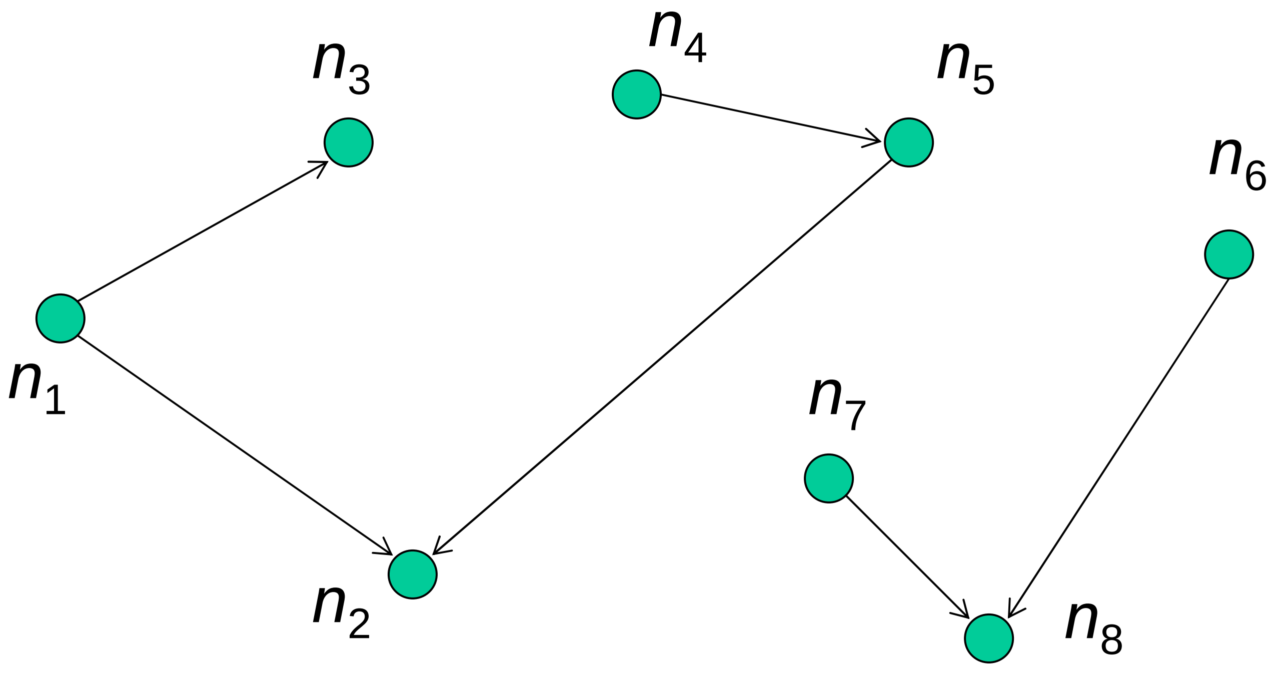
- *Cycle*: Path s.t. start node = end node.
- *Simple cycle*: Cycle without repeated arcs or nodes.
- *Hamiltonian cycle*: A simple forward cycle that contains all the nodes of a the graph.



Red path is a simple cycle

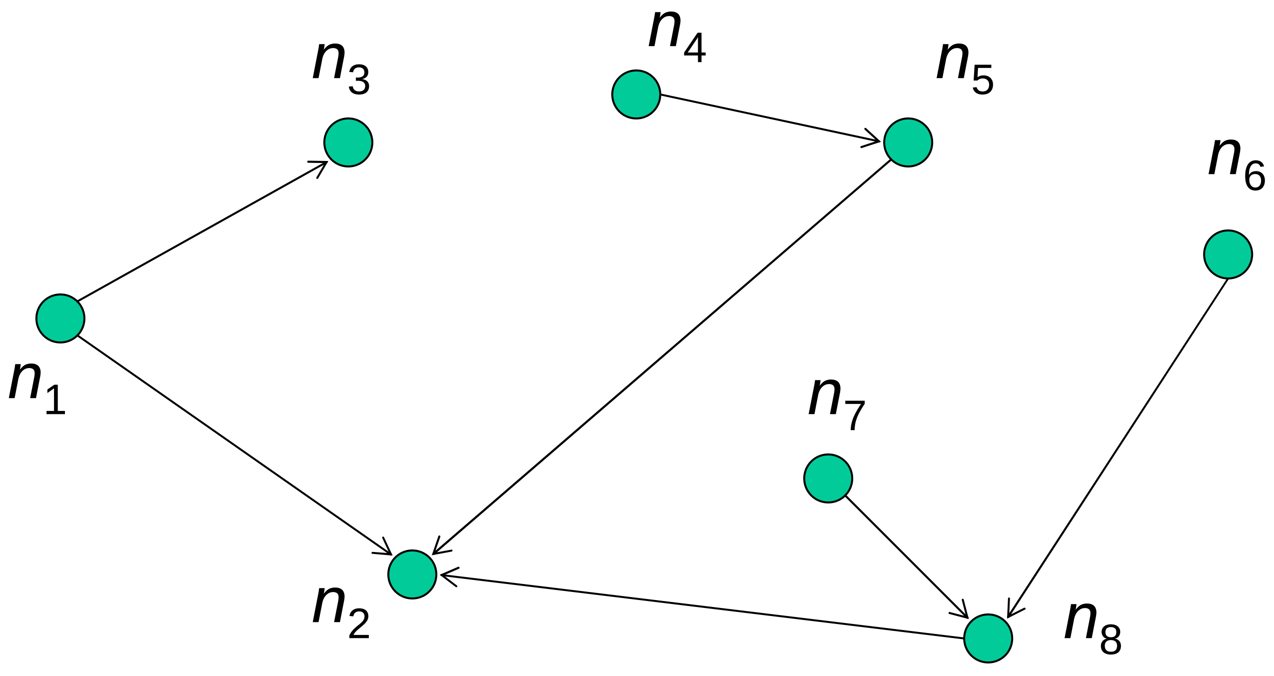
Acyclic graph

- Contains no cycles.

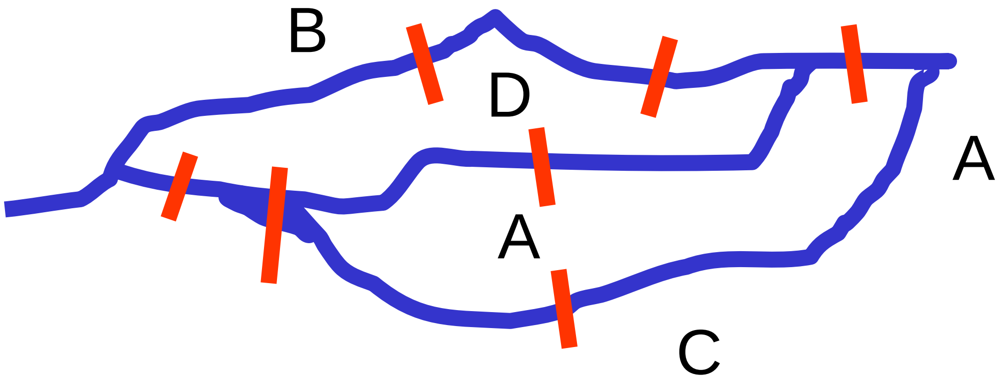


Connected graph, tree

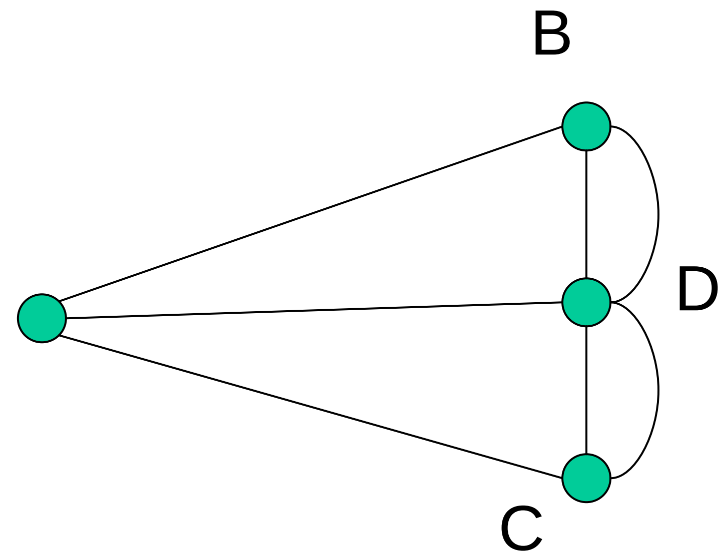
- *Connected* graph: There is a path between any pair of nodes.
- A *tree* is a connected acyclic graph.



Eulerian Graphs



Koenigsberg bridge problem

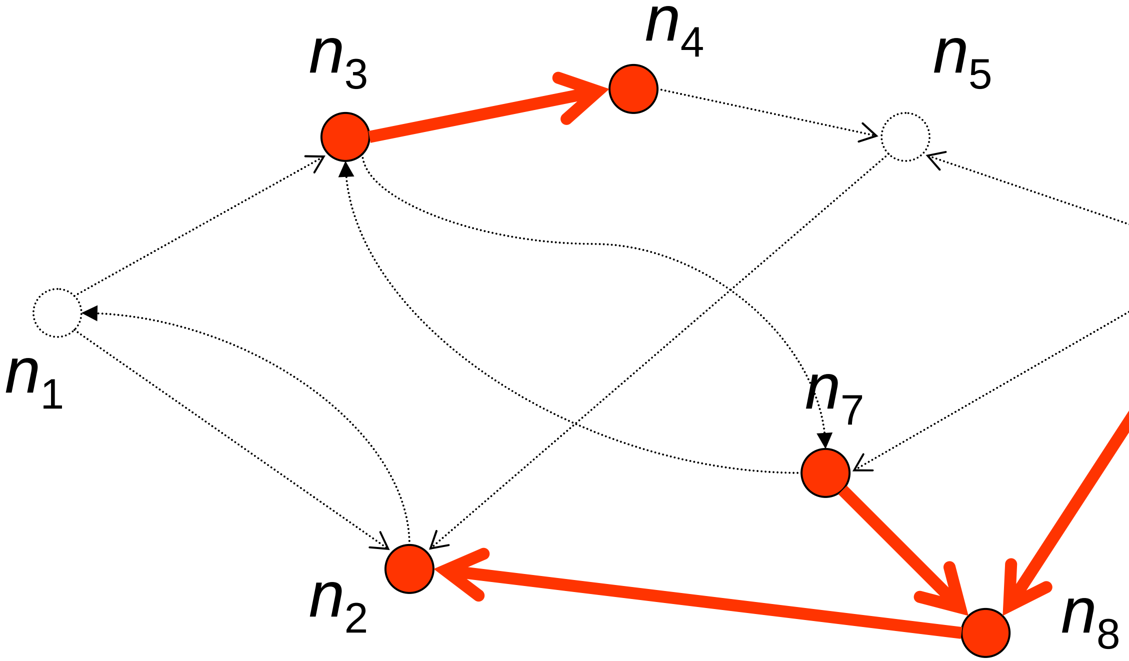


Equivalent graph

- If a graph is connected and each of its nodes has even degree, there is a cycle (not necessarily forward) that contains all the arcs of the cycle exactly once.
- Such a cycle is called an *Euler cycle*.
- Is the Koenigsberg bridge problem solvable?

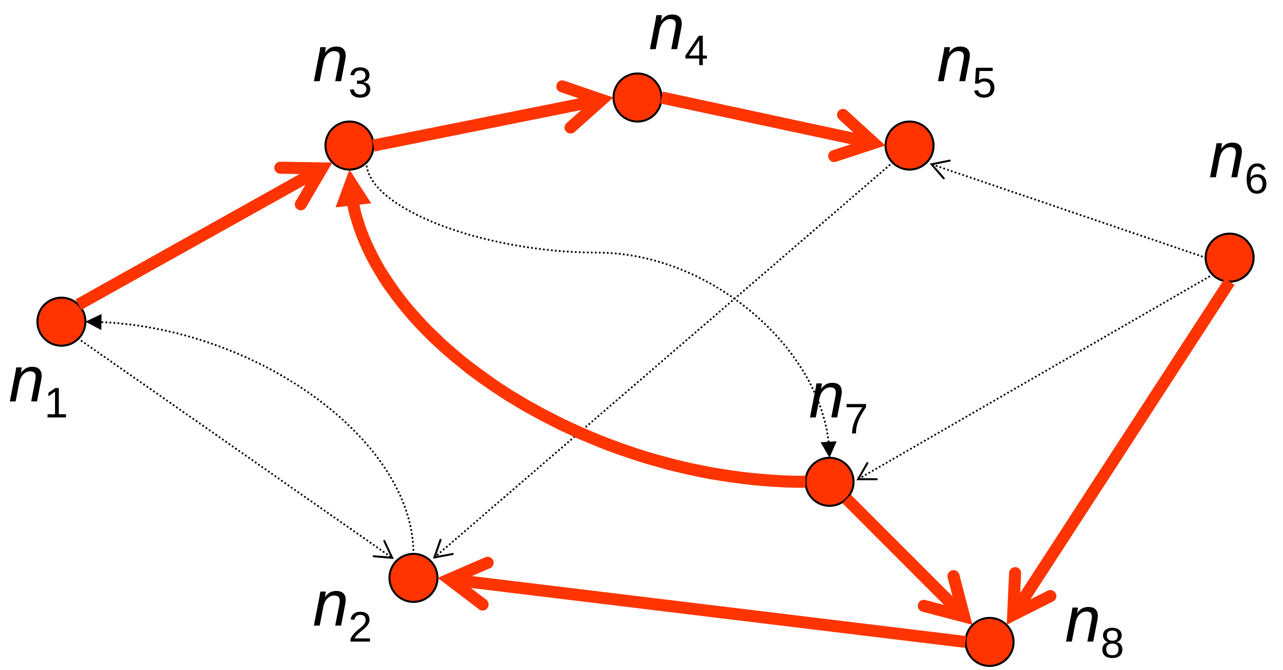
Subgraph

- *Subgraph* $G' = (N', A')$ of graph $G = (N, A)$: $N' \subseteq N$ and $A' \subseteq A$.



Spanning tree

- *Spanning tree* of *G* is a subgraph of G, which is a tree and contains all the nodes of *G*.
- Lemma: A subgraph is a spanning tree *iff* it is connected and contains *N*-1 arcs.

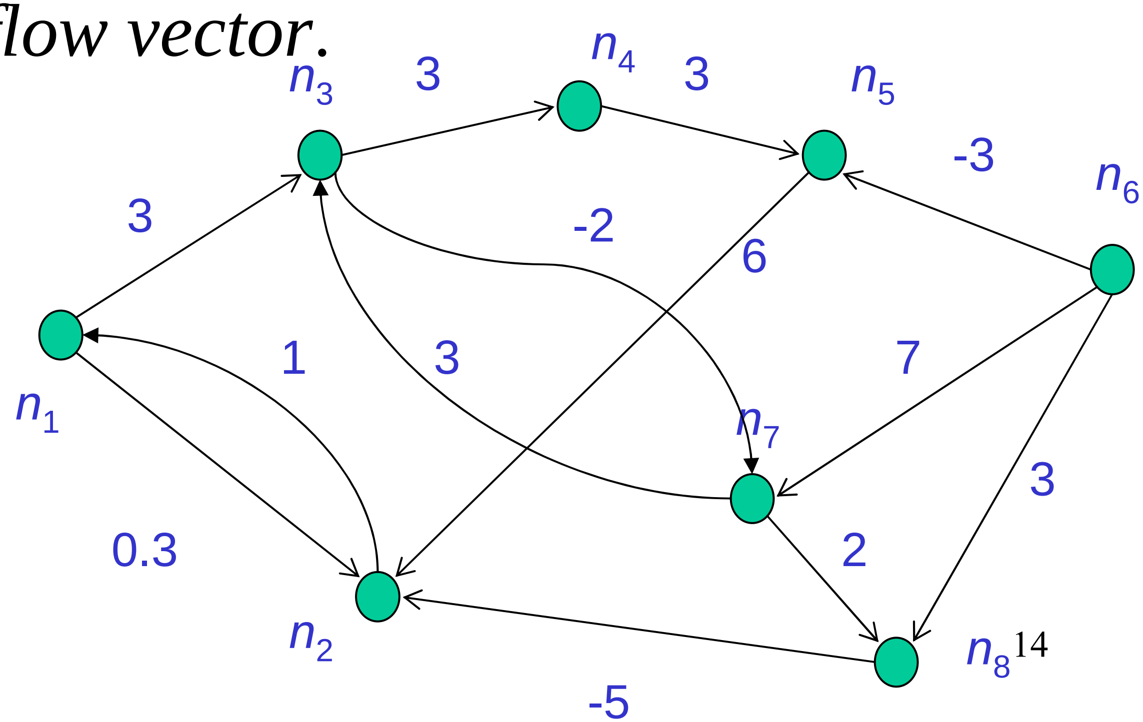


Flow

- *Flow* of arc (i,j):
  A scalar x_{ij} (real, can be negative).
- Given a graph (N,A), a set of flows

$$\{x_{ij} \mid (i,j) \in A\}$$

is referred to as a *flow vector*.

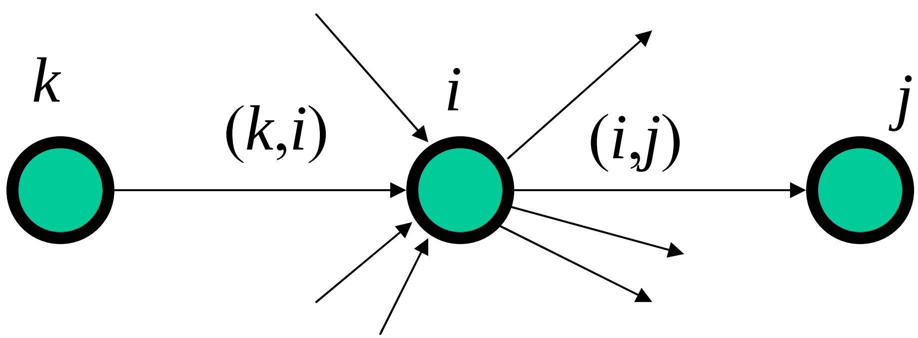


Flow and divergence

- *Divergence* (of i): Given a flow vector x and a graph G

$$y_i = \sum_{\{j|(i,j)\in A\}} x_{ij} - \sum_{\{k|(k,i)\in A\}} x_{ki} \quad \forall i \in N$$

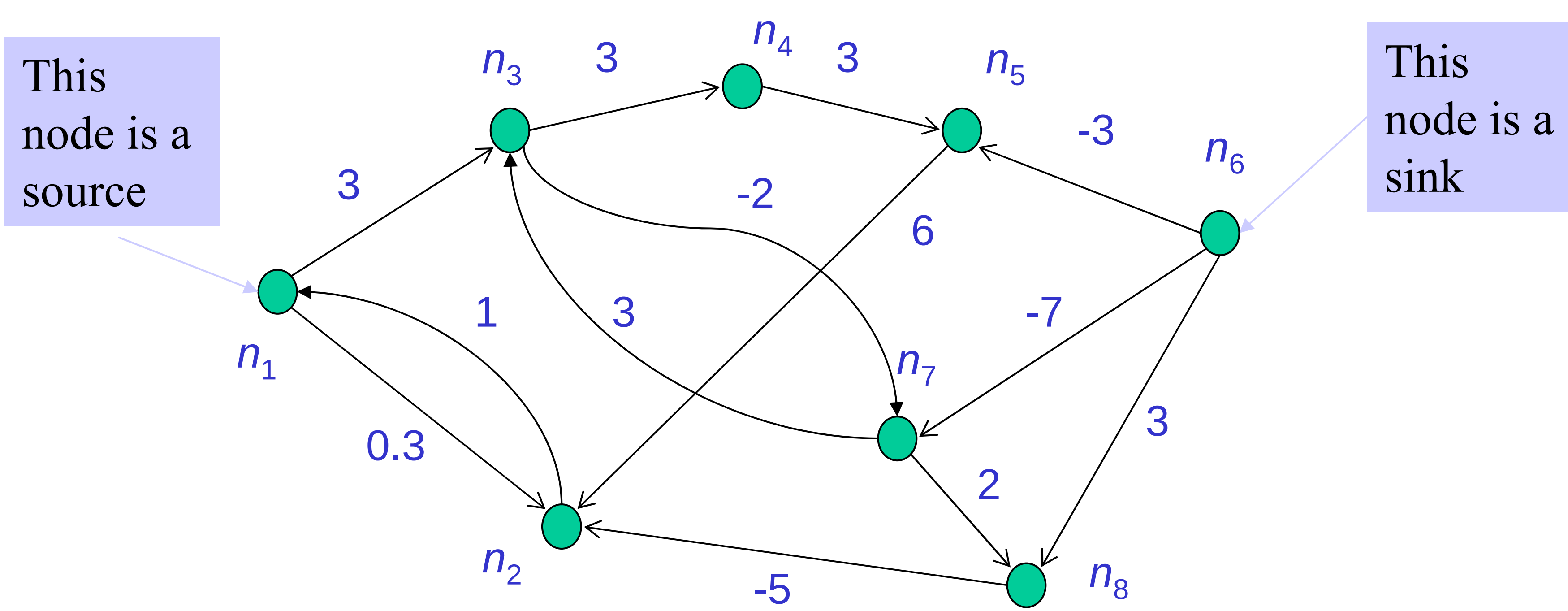
[total flow departing less the total flow arriving]



- Consequence of flow definition: $\sum_{i\in N} y_i = 0$

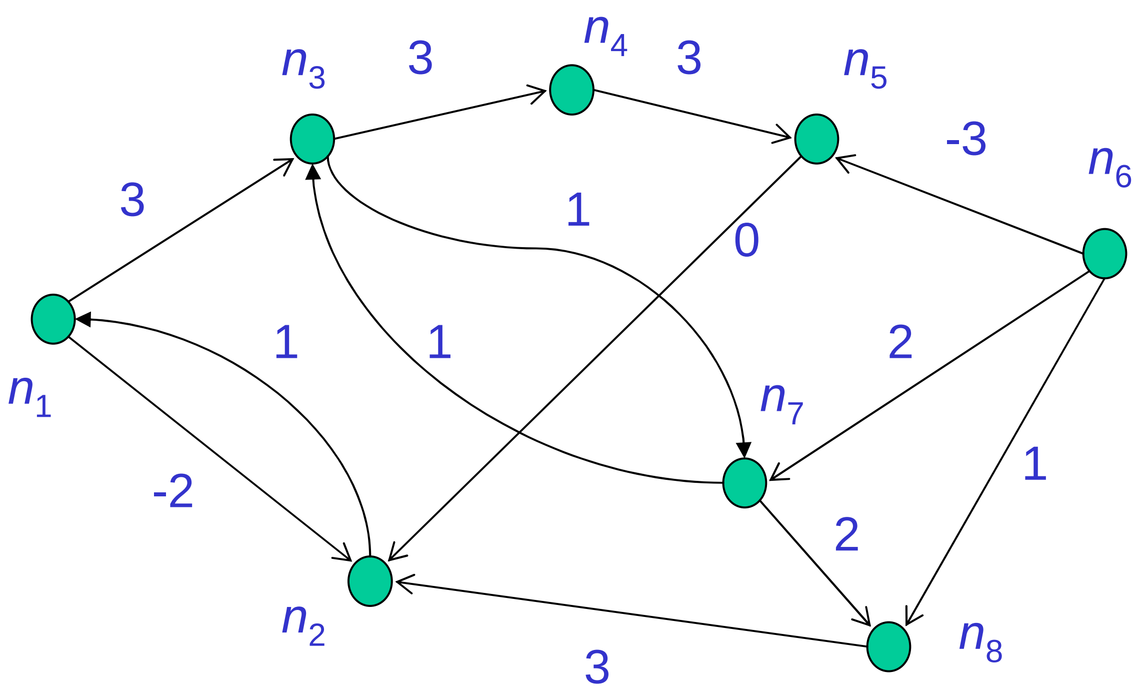
Source/sink

- Node *i* is a *source* (*sink*) for flow x if $y_i > 0$ (<0).



Circulation

- If y_i=0 for all i in N, flow x is called a *circulation*.



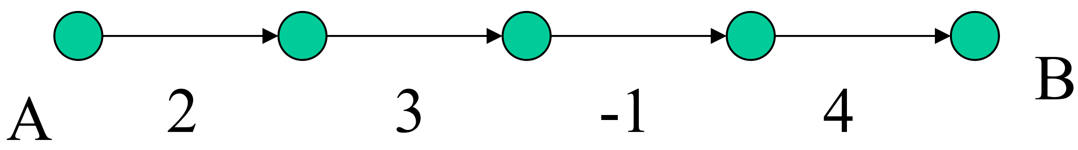
Path P unblocked w.r.t. flow vector x

- Assume that for each arc there is an upper bound c_{ij} to its flow x_{ij}, and a lower bound b_{ij}.
- If $b_{ij}<x_{ij}<c_{ij}$, given a flow vector x, a path P is *unblocked* w.r.t. x iff additional positive flow can be sent along P without violating the constraints, i.e. flow can be increased (decreased) on P^+ (P^-) of the forward (backward) arcs:

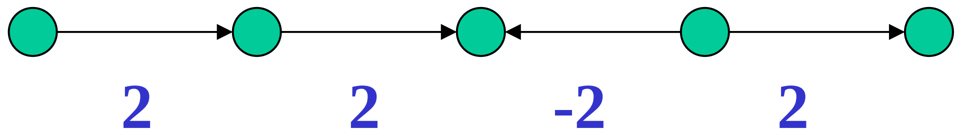
$$x_{ij} < c_{ij} \forall (i, j) \in P^+$$
$$x_{ij} > b_{ij} \forall (i, j) \in P^-$$

Example

- Assuming that in this network c_{ij}=5 and b_{ij}=-5 for all arcs, is the path from A to B blocked?
- What is the maximum additional (positive) flow we can send from A to B?
- Same question from B to A.



Path flows

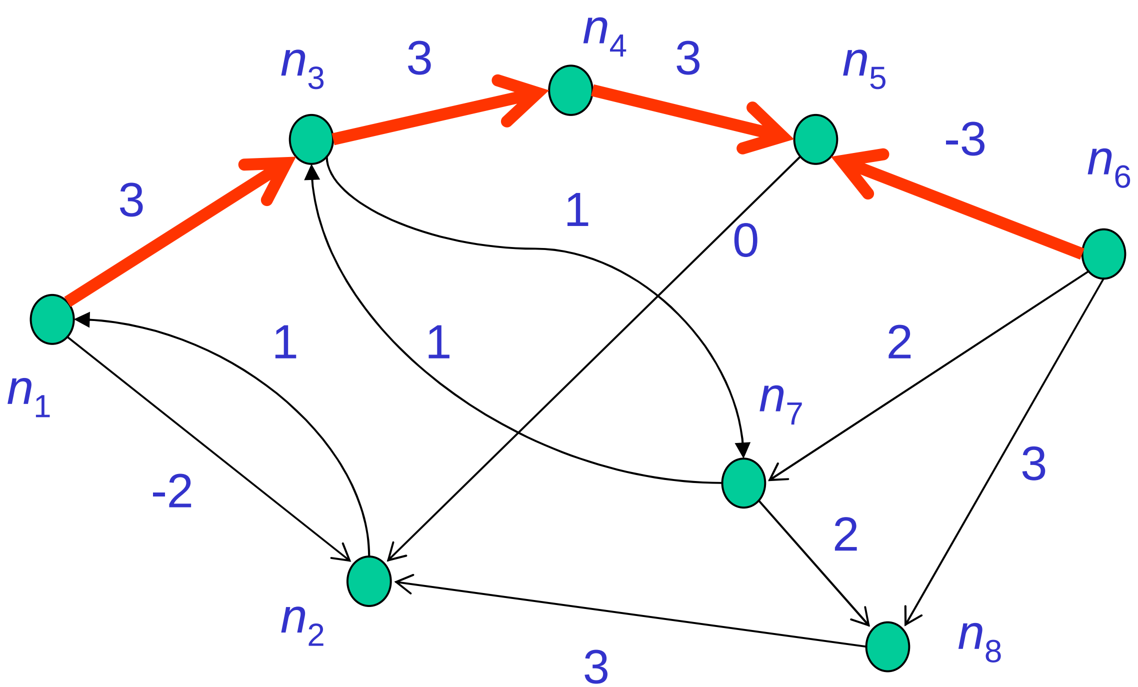


- A *simple path flow* sends a positive amount of flow along a simple path *P*:

$$x_{ij} = \begin{cases} a & if \quad (i,j) \in P^+ \\ -a & if \quad (i,j) \in P^- \\ 0 & otherwise \end{cases}$$

- If *P* is a *cycle*, *x* is called a *simple cycle flow*.
- *P conforms* to *x* if $x_{ij}>0$ for all forward arcs (i,j) of *P* and $xij<0$ for all backward arcs (i,j) of *P* and furthermore *P* is a cycle or else the start node of *P* is a source and the end node of *P* is a sink.

Example



Does the path consisting of the sequence of arcs

$$(n_1,n_3), (n_3,n_4), (n_4,n_5), (n_5,n_6)$$

conform to the flow vector shown?

No, because n_1 is not a source.

Simple path flow conforms to vector flow

- A simple path flow x^s conforms to flow vector x if the path P corresponding to x^s via [eq. in slide 20] conforms to x.
- This is equivalent to

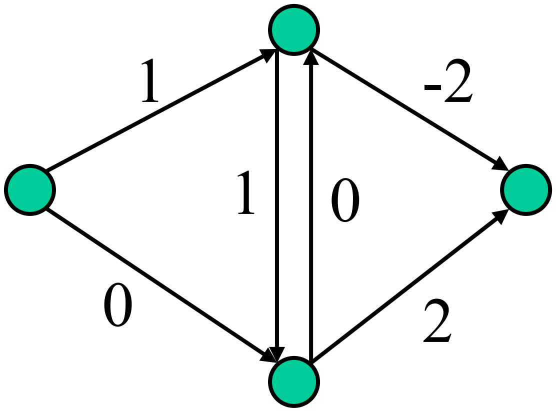
$$0 < x_{ij} \qquad \forall (i, j) : 0 < x_{ij}^s$$

$$x_{ij} < 0 \qquad \forall (i, j) : x_{ij}^s < 0$$

Conformal realization theorem

A nonzero flow vector x can be decomposed into the sum of t simple path flow vectors $x^1, x^2, \ldots, x^t$ that conforms to x, with t being at most equal to the sum of the numbers of arcs and nodes $A+N$. If x is integer, then $x^1, x^2, \ldots, x^t$ can also be chosen to be integer. If x is a circulation, then $x^1, x^2, \ldots, x^t$ can be chosen to be simple cycle flows, and $t \leq A$.

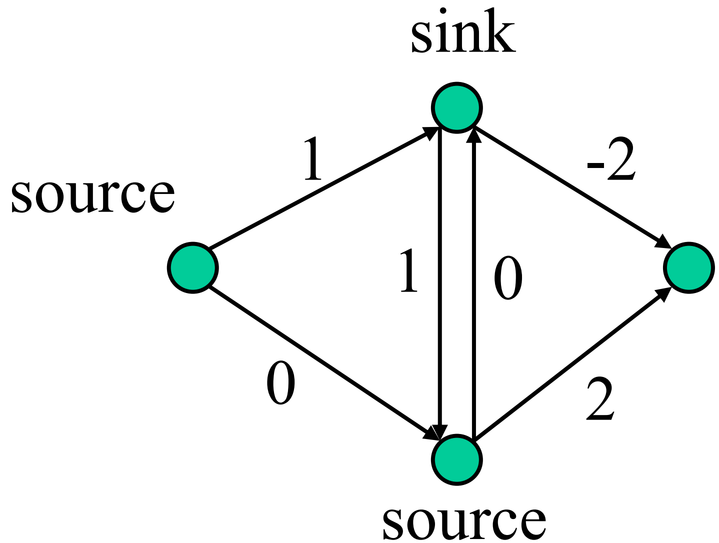
Example



Decompose the vector flow x into simple path flows.

Example (continued)

Step 1: Find sources and sinks.



Step 2: Find simple path flows from sources to sinks or simple cycle flows